Recurrence in Quantum Mechanics

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We first compare the mathematical structure of quantum and classical mechanics when both are formulated in a C*-algebraic framework. By using finite von Neumann algebras, a quantum mechanical analogue of Liouville's theorem is then proposed. We proceed to study Poincaré recurrence in C*-algebras by mimicking the measure theoretic setting. The results are interpreted as recurrence in quantum mechanics, similar to Poincaré recurrence in classical mechanics.

1. INTRODUCTION

The notion of Poincaré recurrence in classical mechanics is quite well-known. Roughly it means that within experimental error a classical system confined to a finite volume in phase space will eventually return to its initial state. This happens because of Liouville's theorem which states that Lebesgue measure is invariant under the Hamiltonian flow.

Recurrence also occurs in quantum mechanics. One approach to recurrence in quantum mechanics has been through the theory of almost periodic functions (see for example, Bocchieri and Loinger, 1957; Hogg and Huberman, 1982; Percival, 1961). Another line of research, involving coherent states, along with possible applications of quantum recurrence, can be traced by Seshadri *et al.* (1999) and references therein. However, these methods differ considerably from the measure theoretic techniques employed to study recurrence in classical mechanics.

In this paper we intend to show how recurrence in quantum mechanics can be cast in a mathematical form that looks the same as the classical case. More precisely, the quantum case is a noncommutative extension of the classical case. Some of the methods presented also provide a general view on how to translate between the quantum and classical descriptions of nature.

A few remarks concerning the mathematical setting are in order. Recently Niculescu *et al.* (in press) working from a purely mathematical viewpoint, showed

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that an analogue of Poincaré recurrence can be obtained in a C*-algebraic framework. Since both quantum and classical mechanics can be formulated in the language of C*-algebras, it seems most natural to work in this setting. In fact, as we shall see in Section 2, quantum mechanics and classical mechanics are identical, except for commutativity, when both are viewed purely in C*-algebraic terms. Our approach to Poincaré recurrence will differ somewhat from that of Niculescu *et al.* (2001) in that we will also consider mappings between C*-algebras, rather than just linear functionals on C*-algebras. Furthermore, instead of looking at arbitrary elements of the algebras, we will concentrate on the projections. The reasons for this will become clear in Sections 2 and 3. The main mathematical results are presented in Section 4.

For these results to have implications for quantum mechanics, we can expect from our remarks concerning the classical case that we will need a quantum mechanical analogue of Liouville's theorem. We propose such an analogue in Section 3, and in the process we are naturally led to consider finite von Neumann algebras. In Section 5 we describe how the theorems of Section 4 would result in recurrence in quantum mechanics. Using the analogy between quantum and classical mechanics we also briefly discuss the properties a quantum mechanical system should most likely have in order to satisfy the requirements of these theorems.

2. QUANTUM MECHANICS AND CLASSICAL MECHANICS IN A C*-ALGEBRAIC SETTING

We start with two simple definitions that apply to both quantum mechanics and classical mechanics:

Definition 2.1. An *observable* of a physical system is any attribute of the system which results in a real number when measured. We call this real number the *value* of the observable during the measurement.

Definition 2.2. Consider any observable of a physical system, and any Borel set $S \subset \mathbb{R}$. We now perform an experiment on the system which results in a "yes" if the value of the observable lies in S during the experiment, and a "no" otherwise; the experiment gives no further information. We call this a *yes/no experiment.*

Definition 2.2 seems justified since in practice there are always experimental errors, in other words we always get a range of values (namely S in Definition 2.2) rather than a single value.

Let's look at the C*-algebraic formulation of quantum mechanics (also see Haag, 1996). Consider any quantum mechanical system. We represent the observables of the system by a unital C*-algebra $\mathfrak{A}$, called the *observable algebra* of the

system, and the state of the system by a state ω on $\mathfrak{A}$ (i.e. ω is a normalized positive linear functional on $\mathfrak{A}$). $\mathfrak{A}$ contains the spectral projections of the system's observables rather than the observables themselves. By this we mean the following: To any yes/no experiment that we can perform on the system, there corresponds a projection P in $\mathfrak{A}$ such that $\omega(P)$ is the probability of getting a "yes" during the experiment for any state ω of the system. We will refer to P as *the* projection of the yes/no experiment.

We will only consider yes/no experiments for which the experimental setup is such that at least in the case of a "yes" the system survives the experiment (for example, it is not absorbed by a detector), so further experiments can be performed on it. What does the system's state look like after such an experiment? Consider for the moment the Hilbert space setting for quantum mechanics. Here the (pure) states of a system are represented by nonzero vectors in a Hilbert space $\mathfrak{H}$, called the *state space* of the system. Suppose the state is given by the unit vector x in $\mathfrak{H}$. After a yes/no experiment the state is given by the projection of x on some Hilbert subspace of $\mathfrak{H}$. Denoting the corresponding projection operator in case of a "yes" by Q, we see that the system's state after the experiment would then be given by the unit vector $Qx/\|Qx\|$. It is clear that Q is the projection of the experiment, since $\|Qx\|^2 = \langle x, Qx \rangle$ is exactly the probability of getting a "yes." (Here the state θ on the C*-algebra $\mathfrak{L}(\mathfrak{H})$ of all bounded linear operators on $\mathfrak{H}$, given by $\theta(A) = \langle x, Ax \rangle$, is the C*-algebraic representation of the state x, in the sense of ω above.)

Returning to our system with observable algebra $\mathfrak{A}$, we know by the GNS-construction (see for example Section 2.3.3 of Bratteli and Robinson, 1987) that there exists a Hilbert space $\mathfrak{H}$, a $*$-homomorphism $\pi : \mathfrak{A} \to \mathfrak{L}(\mathfrak{H})$, and a unit vector Ω in $\mathfrak{H}$, such that

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle \tag{1}$$

for all A in $\mathfrak{A}$. This looks like the usual expression for the expectation value of an observable (here represented by $\pi(A)$) for a system in the state Ω in the Hilbert space setting (compare θ above). On a heuristic level we therefore regard $\mathfrak{H}$ as the state space of the system, and Ω as its state. Say the result of the yes/no experiment with projection P is "yes." On the basis of the Hilbert space setting described earlier, it would now be natural to expect that after the experiment the state is represented by the unit vector $\Omega' = \pi(P)\Omega/\|\pi(P)\Omega\|$, since $\pi(P)$ is the projection of the experiment in the Hilbert space setting in the same way as Q above (and hence $\pi(P)$ here plays the role of Q). Note that $\|\pi(P)\Omega\|^2 = \omega(P) > 0$ since this is exactly the probability of getting the result "yes." We now replace Ω in (1) by Ω' to get a new expectation functional ω' defined by

$$\omega'(A) = \langle \Omega', \pi(A)\Omega' \rangle$$

for all A in $\mathfrak{A}$. Clearly $\omega'(A) = \omega(PAP)/\omega(P)$, so $\omega'(1) = 1$, which implies that ω' is a state on $\mathfrak{A}$. Based on these arguments we give the following postulate:

Postulate 2.3. Consider a quantum mechanical system in the state ω on its observable algebra $\mathfrak{A}$. Suppose we get a "yes" during a yes/no experiment performed on the system. After the experiment the state of the system is then given by the state ω' on $\mathfrak{A}$ defined by

$$\omega'(A) = \omega(PAP)/\omega(P)$$

for all A in $\mathfrak{A}$, where P is the projection of the yes/no experiment.

When expressed in terms of a density operator ρ on a Hilbert space, where $\omega(A) = \text{Tr}(\rho A)$ for a bounded linear operator A on the Hilbert space, this is sometimes refered to as the Lüders rule (see Hughes (1989) or Lüders (1951)).

Lastly we mention that the time-evolution of the system is described by a one-parameter ∗-automorphism group τ of $\mathfrak{A}$, so if the projection of a yes/no experiment is P at time 0, then at time t the projection of the same yes/no experiment will be $\tau_t(P)$.

Now we turn to classical mechanics. We can represent the state of a classical system by a point in its *phase space* $\mathbb{R}^{2n}$. This is somewhat restrictive since such a point represents exact knowledge of the state of the system, which is impossible in practice. Therefore, we rather represent the state of the system by a Borel measure μ on $\mathbb{R}^{2n}$ such that $\mu(S)$ is the probability that the system's state is a point somewhere in the Borel set $S \subset \mathbb{R}^{2n}$. In particular we have $\mu(\mathbb{R}^{2n}) = 1$.

We view each observable of the system as a Borel function $f : \mathbb{R}^{2n} \to \mathbb{R}$. This simply means that if the system's state is the point x in $\mathbb{R}^{2n}$, then the value of the observable is $f(x)$. If we perform a yes/no experiment to determine if f's value lies in the Borel set $S \subset \mathbb{R}$, then the probability of getting "yes" is clearly

$$\mu(f^{-1}(S)) = \int \chi_{f^{-1}(S)} \, d\mu$$

where χ denotes characteristic functions (i.e. for any set A, the function χ_A assumes the value 1 on A, and zero everywhere else). We can view $\chi_{f^{-1}(S)}$ as a spectral projection of the observable f, and we will refer to it as *the* projection of the yes/no experiment, just as in the quantum mechanical case. Note that $\chi_{f^{-1}(S)}$ is a projection in the C*-algebra $B_\infty(\mathbb{R}^{2n})$ of all bounded complex-valued Borel functions on $\mathbb{R}^{2n}$. We can define a state ω on the C*-algebra $B_\infty(\mathbb{R}^{2n})$ by

$$\omega(g) = \int g \, d\mu$$

for all g in $B_\infty(\mathbb{R}^{2n})$. Then we see that the probability of getting a "yes" in the above mentioned yes/no experiment is $\omega(\chi_{f^{-1}(S)})$. So we can view ω as representing the

state of the system in exactly the same way as in quantum mechanics, where now $B_\infty(\mathbb{R}^{2n})$ is the unital C*-algebra representing the observables of the system. For this reason we call $B_\infty(\mathbb{R}^{2n})$ the *observable algebra* of the system. Postulate 2.3 then holds for the classical case as well since a "yes" will mean the system's state is a point in $f^{-1}(S)$, in which case we can describe the system's state after the experiment by the measure μ' given by

$$\mu'(V) = \mu(V \cap f^{-1}(S))/\mu(f^{-1}(S))$$

for all Borel sets $V \subset \mathbb{R}^{2n}$. As in the case of μ and ω above, μ' corresponds to the state ω' on $B_\infty(\mathbb{R}^{2n})$ given by

$$\omega'(g) = \int g \, d\mu' = \omega\big(\chi_{f^{-1}(S)} g \chi_{f^{-1}(S)}\big)/\omega\big(\chi_{f^{-1}(S)}\big)$$

(the second equality follows using standard measure theoretic arguments, i.e. first prove it for g a characteristic function and then use Lebesgue convergence). This is exactly what Postulate 2.3 says if we replace the word "quantum" by "classical."

For the time-evolution of a classical system we need the concept of a flow. Consider a measure space (X, Σ, μ), where μ is a measure defined on a σ-algebra Σ of subsets of the set X. A *flow* on (X, Σ, μ) is a mapping $t \mapsto T_t$ on $\mathbb{R}$ with the following properties: T_t is a function defined on X to itself, T_0 is the identity on X (i.e. $T_0(x) = x$), $T_s \circ T_t = T_{s+t}$, and $T_t(S) \in \Sigma$ and $\mu(T_t(S)) = \mu(S)$ for all S in Σ. We denote this flow simply by T_t.

The time-evolution of our classical system is given by a flow T_t on $(\mathbb{R}^{2n}, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the σ-algebra of Borel sets of $\mathbb{R}^{2n}$, and λ is the Lebesgue measure on $\mathbb{R}^{2n}$. Note that this statement contains Liouville's theorem, namely $\lambda(T_t(S)) = \lambda(S)$ for all S in $\mathcal{B}$. We call T_t the *Hamiltonian flow*. It simply means that if at time 0 the system is in the state $x \in \mathbb{R}^{2n}$, then at time t it is in the state $T_t(x)$.

As in the C*-algebraic approach to quantum mechanics, we want the time-evolution to act on the observable algebra rather than on the states. It is clear that an observable given by f at time 0, will then be given by $f \circ T_t$ at time t (the well-known Koopman construction; Koopman, 1931). This is equivalent to the action of T_t on the spectral projections of f, since $\chi_{(f \circ T_t)^{-1}(S)} = \chi_{f^{-1}(S)} \circ T_t$ for all Borel sets $S \subset \mathbb{R}$. It is easily seen that if we define τ by

$$\tau_t(g) = g \circ T_t \qquad (2)$$

for all g in $B_\infty(\mathbb{R}^{2n})$, then τ is a one-parameter *-automorphism group of the C*-algebra $B_\infty(\mathbb{R}^{2n})$. So the time-evolution is described in exactly the same way as in quantum mechanics when we are working in the C*-algebraic setting.

We have now obtained a C*-algebraic formulation of classical mechanics. Note that $B_\infty(\mathbb{R}^{2n})$ is an Abelian C*-algebra. Replacing $B_\infty(\mathbb{R}^{2n})$ by an arbitrary Abelian unital C*-algebra would give us an abstract C*-algebraic formulation of classical mechanics. From our discussion above it is clear that if in the C*-algebraic

formulation of quantum mechanics described earlier we assume that $\mathfrak{A}$ is Abelian, then we get exactly this abstract C*-algebraic formulation of classical mechanics. Setting $\mathfrak{A} = B_\infty(\mathbb{R}^{2n})$ would make it concrete. In this sense the C*-algebraic formulation of quantum mechanics actually contains classical mechanics as a special case.

3. A QUANTUM MECHANICAL ANALOGUE OF LIOUVILLE'S THEOREM

We have seen in Section 2 that in purely C*-algebraic terms quantum mechanics and classical mechanics are identical, except of course for the fact that the classical observable algebra is Abelian while this is not in general true for quantum mechanics. This suggests that it might be possible to find a quantum mechanical analogue of Liouville's theorem. Our first clue in this direction is the following simple proposition, which is proved by standard measure theoretic arguments:

Proposition 3.1. *Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$, and let $T : X \to X$ be a mapping such that $T^{-1}(S) \in \Sigma$ for all $S \in \Sigma$. Let $B_\infty(\Sigma)$ be the C*-algebra of all bounded complex-valued Σ-measurable functions on X, and define τ and φ by $\tau(g) = g \circ T$ and $\varphi(g) = \int g \, d\mu$ for all $g \in B_\infty(\Sigma)$. Then $\mu(T^{-1}(S)) = \mu(S)$ for all $S \in \Sigma$ if and only if $\varphi(\tau(g)) = \varphi(g)$ for all $g \in B_\infty(\Sigma)$.*

Consider a classical system confined to a bounded Borel set F in the phase space $\mathbb{R}^{2n}$. So $\lambda(F) < \infty$, where λ is the Lebesgue measure on $\mathbb{R}^{2n}$. We define a measure ν on the Borel sets of $\mathbb{R}^{2n}$ by

$$\nu(S) = \lambda(S \cap F).$$

Using Proposition 3.1 we see that Liouville's theorem for this system can then be expressed in C*-algebraic terms by stating that

$$\varphi(\tau_t(g)) = \varphi(g) \tag{3}$$

for all g in $B_\infty(\mathbb{R}^{2n})$, where τ is given by (2), and $\varphi(g) = \int g \, d\nu$ (so φ is a positive linear functional on $B_\infty(\mathbb{R}^{2n})$). Note that the condition $\mu(X) < \infty$ in Proposition 3.1 can be dropped if we only consider positive elements of $B_\infty(\Sigma)$. Hence (3) would express Liouville's theorem for systems not necessarily bounded in phase space if we were to use λ instead of ν, and only consider positive elements g of $B_\infty(\mathbb{R}^{2n})$. (In this case φ could assume infinite values and it would not be a linear mapping on $B_\infty(\mathbb{R}^{2n})$ any more.) We only work with the bounded case in recurrence though.

Because of Section 2, we now suspect that a quantum mechanical analogue of Liouville's theorem should have the same form as (3). Let's look at this from a different angle. In the Hilbert space setting for quantum mechanics, the state space $\mathfrak{H}$ can be viewed as the analogue of the classical phase space $\mathbb{R}^{2n}$. $\mathfrak{H}$ is a Hilbert space while we view $\mathbb{R}^{2n}$ purely as a measure space. Apart from dynamics, we saw in Section 2 that the central objects in both quantum and classical mechanics are the projections. A projection defined on $\mathfrak{H}$ is equivalent to a Hilbert subspace of $\mathfrak{H}$ (namely the range of the projection). A projection defined on $\mathbb{R}^{2n}$ is a Borel measurable characteristic function, and is therefore equivalent to a Borel set in $\mathbb{R}^{2n}$. Liouville's theorem is based on the existence of a natural way of measuring the size of a Borel set in $\mathbb{R}^{2n}$, namely the Lebesgue measure λ. We would therefore like to have a natural way of measuring the size of a Hilbert subspace of $\mathfrak{H}$ to get a quantum analogue of Liouville's theorem. An obvious candidate is the (Hilbert) dimension dim. For the Hamiltonian flow T_t, Liouville's theorem states that $\lambda(T_{-t}(S)) = \lambda(S)$ for every Borel set S. (We use $T_{-t}(S)$ instead of $T_t(S)$, since this corresponds to the action of T_t on the observable algebra rather than on the states, namely $\chi_S \circ T_t = \chi_{T_{-t}(S)}$.) In the state space time-evolution is given by a one-parameter unitary group U_t on $\mathfrak{H}$, and for any Hilbert subspace $\mathfrak{K}$ of $\mathfrak{H}$ we have $\dim(U_t^* \mathfrak{K}) = \dim(U_{-t}\mathfrak{K}) = \dim(\mathfrak{K})$. This is clearly similar to Liouville's theorem. For a finite dimensional state space we will in fact view this as a quantum analogue of Liouville's theorem. However, since state spaces are usually infinite dimensional, we would like to work with something similar to Hilbert dimension, which does not assume infinite values.

This leads us naturally to the C*-algebras known as finite von Neumann algebras (see for example Kadison and Ringrose, 1986), since for such an algebra there is a dimension function, defined on the projections of the algebra, which does not assume infinite values. This function is in fact the restriction of a so-called trace defined on the whole algebra, so we might as well work with this trace. We now explain this in more detail.

Let $\mathfrak{M}$ be a finite von Neumann algebra on a Hilbert space $\mathfrak{H}$, and let $\mathfrak{M}'$ be its commutant. Then there is a unique positive linear mapping tr: $\mathfrak{M} \to \mathfrak{M} \cap \mathfrak{M}'$ such that $\text{tr}(AB) = \text{tr}(BA)$ and $\text{tr}(C) = C$ for all A, $B \in \mathfrak{M}$ and $C \in \mathfrak{M} \cap \mathfrak{M}'$. We call tr the *trace* of $\mathfrak{M}$. We mention that in the special case where $\mathfrak{M} = \mathfrak{L}(\mathfrak{H})$, with $\mathfrak{H}$ finite dimensional, tr is just the usual trace (sum of eigenvalues) normalized such that $\text{tr}(1) = 1$.

For a projection $P \in \mathfrak{M}$ of $\mathfrak{H}$ onto the Hilbert subspace $\mathfrak{K}$, we see that $U_t^* P U_t$ is the projection of $\mathfrak{H}$ onto $U_t^* \mathfrak{K}$, where U_t is a one-parameter unitary group on $\mathfrak{H}$. So in the framework of finite von Neumann algebras we would like to replace the equation $\dim(U_t^* \mathfrak{K}) = \dim(\mathfrak{K})$ mentioned above by $\text{tr}(U_t^* P U_t) = \text{tr}(P)$.

If a self-adjoint (possibly unbounded) operator A in $\mathfrak{H}$ is an observable and $\mathfrak{M}$ an observable algebra of a physical system, then we want the spectral projections $\chi_S(A)$ of A to be contained in $\mathfrak{M}$, where S is any Borel set in $\mathbb{R}$, since these

projections are the projections of the yes/no experiments that can be performed on the system. But then $f(A) \in \mathfrak{M}$ for any bounded complex-valued Borel function f on $\mathbb{R}$. In particular $e^{-iAt} \in \mathfrak{M}$ for all real t.

For these reasons we will consider physical systems of the following nature:

Definition 3.2. A *bounded quantum system* is a quantum mechanical system for which we can take the observable algebra as a finite von Neumann algebra $\mathfrak{M}$ on a Hilbert space $\mathfrak{H}$ such that the Hamiltonian H of the system is a self-adjoint (possibly unbounded) operator in $\mathfrak{H}$ with $e^{-iHt} \in \mathfrak{M}$ for real t. We denote this system by $(\mathfrak{M}, \mathfrak{H}, H)$.

The reason for the term "bounded" will become clear in Section 5. We now propose a quantum analogue of Liouville's theorem based on the intuitive arguments in terms of dimension given above. We give it in the form of a proposition:

Proposition 3.3. *Consider a bounded quantum system* $(\mathfrak{M}, \mathfrak{H}, H)$. *Then* $U_t = e^{-iHt}$ *is a one-parameter unitary group on* $\mathfrak{H}$. *Let* τ *be the time-evolution of the system, i.e.* $\tau_t(A) = U_t^* A U_t$ *for all* $A \in \mathfrak{M}$. *Then*

$$tr(\tau_t(A)) = tr(A) \qquad (4)$$

for all A *in* $\mathfrak{M}$, *where tr is the trace of* $\mathfrak{M}$. *(This last statement is our quantum analogue of Liouville's theorem.)*

Proof: Since $U_t \in \mathfrak{M}$, we have $\mathrm{tr}(\tau_t(A)) = \mathrm{tr}(U_t^* A U_t) = \mathrm{tr}(U_t U_t^* A) = \mathrm{tr}(A)$.
□

As we suspected, our quantum analogue of Liouville's theorem, expressed by (4), is of the same form as the C*-algebraic formulation of the classical Liouville theorem as given by (3), with φ replaced by tr. Remember that φ and tr are both positive linear mappings on the respective observable algebras.

Remark. The classical Liouville theorem can also be expressed in terms of the Liouville equation

$$\frac{\partial \rho}{\partial t} = \{\rho, H\}$$

where $\rho : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$ is the density function, H the classical Hamiltonian, and $\{\cdot, \cdot\}$ the Poisson bracket. This equation can be seen as describing the flow of a fluid in phase space such that at any point moving along with the fluid, the density of the fluid remains constant. So besides giving the time-evolution, this equation also

states a property of the time-evolution, namely that it conserves volume in phase space. In quantum mechanics we have the analogous von Neumann equation

$$\frac{d\rho}{dt} = i[\rho, H]$$

where $\rho : \mathbb{R} \to \mathfrak{L}(\mathfrak{H})$ is the density operator as a function of time (note that here the derivative with respect to time is total instead of partial). This equation merely gives the time-evolution $\rho(t) = \tau_{-t}(\rho(0))$ of the density operator, where τ is the time-evolution on the observable algebra here viewed as acting on the state instead of the observables. Von Neumann's equation by itself should therefore not be regarded as a quantum mechanical analogue of Liouville's theorem.

4. POINCARÉ RECURRENCE IN C*-ALGEBRAS

In Section 3, we proposed a quantum analogue of Liouville's theorem for bounded quantum systems. So, by analogy with classical mechanics, these are the type of systems for which we could expect recurrence. In this section, however, we will be able to study Poincaré recurrence in the more general setting of abstract C*-algebras.

As we shall see, the theory is surprisingly close to the usual measure theoretic setting. It therefore seems appropriate to briefly review Poincaré's recurrence theorem and its proof. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the positive integers. Consider a measure space (X, Σ, μ) with $\mu(X) < \infty$, and let $T : X \to X$ be a mapping such that $\mu(T^{-1}(S)) = \mu(S)$ for all S in Σ. This is merely an abstraction of Liouville's theorem. For some $S \in \Sigma$, suppose that $\mu(S \cap T^{-n}(S)) = 0$ for all $n \in \mathbb{N}$. For all $n, k \in \mathbb{N}$ we then have $\mu(T^{-k}(S) \cap T^{-(n+k)}(S)) = \mu(T^{-k}(S \cap T^{-n}(S))) = \mu(S \cap T^{-n}(S)) = 0$. So $\mu(T^{-m}(S) \cap T^{-n}(S)) = 0$ for all $m, n \in \mathbb{N}$ with $m \neq n$. It follows that

$$\mu(X) \geq \mu\left(\bigcup_{k=1}^{n} T^{-k}(S)\right) = \sum_{k=1}^{n} \mu(T^{-k}(S)) = \sum_{k=1}^{n} \mu(S) = n\mu(S).$$

Letting $n \to \infty$ it follows that $\mu(S) = 0$. This is one form of Poincaré's recurrence theorem, namely if $\mu(S) > 0$, then there exists a positive integer n such that $\mu(S \cap T^{-n}(S)) > 0$. It tells us that S contains a set $S \cap T^{-n}(S)$ of positive measure, which is mapped back into S by T^n.

Note that the mapping $g \mapsto \tau(g) = g \circ T$ is a *-homomorphism of the C*-algebra $B_\infty(\Sigma)$ into itself such that $\varphi(\tau(g)) = \varphi(g)$ and $\mu(S \cap T^{-n}(S)) = \varphi(\chi_S \tau^n(\chi_S))$ for $S \in \Sigma$, where $\varphi(g) = \int g \, d\mu$ for all $g \in B_\infty(\Sigma)$. Using this notation Poincaré's recurrence theorem can be stated as follows: If $\varphi(\chi_S) > 0$, then there exists a positive integer n such that $\varphi(\chi_S \tau^n(\chi_S)) > 0$. The general C*-algebraic approach will now be modelled after this situation. We also get some inspiration from Postulate 2.3, for reasons to be explained in Section 5.

Definition 4.1. Let $\mathfrak{A}$ be a $*$-algebra, and $\mathfrak{B}$ a unital C*-algebra. Let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a positive mapping (i.e. $\varphi(A^*A) \geq 0$ for all $A \in \mathfrak{A}$). We call φ *additive* if

$$\sum_{k=1}^{n} \varphi(P_k) \leq 1$$

for any projections $P_1, \ldots, P_n \in \mathfrak{A}$ for which $\varphi(P_k P_l P_k) = 0$ if $k < l$. We call φ *faithful* if it is linear, $\mathfrak{A}$ is unital, $\varphi(1) = 1$, and $\varphi(A^*A) > 0$ for all nonzero A in $\mathfrak{A}$. We call φ a C*-*trace* if it is linear, $\mathfrak{A}$ is unital, $\varphi(1) = 1$, and for all $A, B \in \mathfrak{A}$ we have $\varphi(AB) = \varphi(BA)$. (Remember: Any C*-algebra is a $*$-algebra.)

If the positive mapping φ given in Definition 4.1 is faithful, then it is also additive, as we now show. Let $P_1, \ldots, P_n \in \mathfrak{A}$ be any projections for which $\varphi(P_k P_l P_k) = 0$ if $k < l$. For $k < l$ we then have $\varphi((P_l P_k)^* P_l P_k) = 0$, so $P_l P_k = 0$, and therefore $P_k P_l = (P_l P_k)^* = 0$. This implies that

$$\sum_{k=1}^{n} P_k \leq 1$$

since the left-hand side is a projection in $\mathfrak{A}$. Thus

$$\sum_{k=1}^{n} \varphi(P_k) = \varphi\left(\sum_{k=1}^{n} P_k\right) \leq \varphi(1) = 1$$

as promised.

In the measure theoretic setting described earlier, we can assume without loss of generality that $\mu(X) = 1$. Then $\varphi : B_\infty(\Sigma) \to \mathbb{C}$ is an additive C*-trace since

$$\sum_{k=1}^{n} \varphi(\chi_{S_k}) = \sum_{k=1}^{n} \mu(S_k) = \mu\left(\bigcup_{k=1}^{n} S_k\right) \leq \mu(X)$$

for any $S_1, \ldots, S_n \in \Sigma$ such that $\varphi(\chi_{S_k} \chi_{S_l}) = \mu(S_k \cap S_l) = 0$ if $k \neq l$.

We now state and prove a C*-algebraic version of Poincaré's recurrence theorem:

Theorem 4.2. *Consider a $*$-algebra $\mathfrak{A}$ and a unital C*-algebra $\mathfrak{B}$, and let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be an additive mapping. Let $\tau : \mathfrak{A} \to \mathfrak{A}$ be a $*$-homomorphism such that $\varphi(\tau(PQP)) = \varphi(PQP)$ for all projections $P, Q \in \mathfrak{A}$. Then, for any projection $P \in \mathfrak{A}$ such that $\varphi(P) > 0$, there exists a positive integer n such that $\varphi(P\tau^n(P)P) > 0$.*

Proof: Note that $\varphi(P\tau^n(P)P) = \varphi((\tau^n(P)P)^* \tau^n(P)P) \geq 0$ for all $n \in \mathbb{N}$. We now imitate the measure theoretic proof.

Suppose $\varphi(P\tau^n(P)P) = 0$ for all $n \in \mathbb{N}$. For all $k, n \in \mathbb{N}$ we then have

$$\varphi(\tau^k(P)\tau^{n+k}(P)\tau^k(P)) = \varphi(\tau^k(P\tau^n(P)P)) = \varphi(P\tau^n(P)P) = 0.$$

Since φ is additive, it follows for any $n \in \mathbb{N}$ that

$$\sum_{k=1}^{n} \varphi(\tau^k(P)) \leq 1.$$

Furthermore,

$$\sum_{k=1}^{n} \varphi(\tau^k(P)) = \sum_{k=1}^{n} \varphi(P) = n\varphi(P) \geq 0$$

since φ is positive and $P = P^*P$. Hence $0 \leq n\varphi(P) \leq 1$, and therefore $n\|\varphi(P)\| \leq 1$. Letting $n \to \infty$, it follows that $\varphi(P) = 0$. $\square$

It is clear that the measure theoretic Poincaré recurrence theorem stated above is just a special case of Theorem 4.2, since the projections of the C*-algebra $B_\infty(\Sigma)$ are exactly the characteristic functions χ_S, where $S \in \Sigma$.

Note that the trace of a finite von Neumann algebra is a faithful C*-trace, hence we have the following corollary of Theorem 4.2, which will be used in Section 5:

Corollary 4.3. *Consider a finite von Neumann algebra $\mathfrak{M}$, and let tr be its trace. Let $\tau : \mathfrak{M} \to \mathfrak{M}$ be a *-homomorphism such that $tr(\tau(A)) = tr(A)$ for all A in $\mathfrak{M}$. Then, for any projection $P \in \mathfrak{M}$ such that $tr(P) > 0$, there exists a positive integer n such that $tr(P\tau^n(P)) > 0$.*

We can also give a C*-algebraic version of Khintchine's theorem (see Petersen (1983), for example, as well as Niculescu *et al.* (in press)). But first we mention that a subset E of $\mathbb{N}$ is called *relatively dense* in $\mathbb{N}$ if there is an $n \in \mathbb{N}$ such that the set

$$E \cap \{j, j+1, \ldots, j+n-1\}$$

is nonvoid for every $j \in \mathbb{N}$.

Theorem 4.4. *Consider a unital C*-algebra $\mathfrak{A}$, and let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a C*-trace. Let $\tau : \mathfrak{A} \to \mathfrak{A}$ be a *-homomorphism such that $\tau(1) = 1$ and $\varphi(\tau(A^*A)) \leq \varphi(A^*A)$ for every A in $\mathfrak{A}$. For any projection P in $\mathfrak{A}$, and any $\varepsilon > 0$, it then follows that the set*

$$E = \{k \in \mathbb{N} : \varphi(P\tau^k(P)) > \varphi(P)^2 - \varepsilon\}$$

is relatively dense in $\mathbb{N}$.

Proof: Let $(\mathfrak{H}, \pi, \Omega)$ be the cyclic representation of $\mathfrak{A}$ obtained from φ by the GNS-construction (just as from ω in (1)). This gives us a linear function

$$\iota : \mathfrak{A} \to \mathfrak{H} : A \mapsto \pi(A)\Omega$$

such that $\iota(\mathfrak{A})$ is dense in $\mathfrak{H}$, and

$$\langle \iota(A), \iota(B) \rangle = \langle \Omega, \pi(A^*B)\Omega \rangle = \varphi(A^*B)$$

for all $A, B \in \mathfrak{A}$. For any $A \in \mathfrak{A}$ we therefore have

$$\|\iota(\tau(A))\|^2 = \varphi((\tau(A))^*\tau(A)) = \varphi(\tau(A^*A))$$
$$\leq \varphi(A^*A)$$
$$= \|\iota(A)\|^2.$$

By the linearity of ι and τ it now follows that

$$\bar{\tau} : \iota(\mathfrak{A}) \to \mathfrak{H} : \iota(A) \mapsto \iota(\tau(A))$$

is well defined (namely if $\iota(A) = \iota(B)$, then $\iota(\tau(A)) = \iota(\tau(B))$), linear and bounded, with $\|\bar{\tau}\| \leq 1$. Since $\bar{\tau}$ is bounded, we can extend it linearly to the whole of $\mathfrak{H}$, keeping $\|\bar{\tau}\| \leq 1$. We are now in a position to imitate the proof of the measure theoretic Khintchine theorem.

Let Q be the projection of $\mathfrak{H}$ onto $\{x \in \mathfrak{H} : \bar{\tau}x = x\}$. For $k \in \mathbb{N}$ we have

$$\varphi(P\tau^k(P)) = \langle \iota(P), \iota(\tau^k(P)) \rangle = \langle \iota(P), \bar{\tau}^k \iota(P) \rangle = \langle x, \bar{\tau}^k x \rangle$$

where $x = \iota(P)$. By the mean ergodic theorem we know that there exists an $n \in \mathbb{N}$ such that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} \bar{\tau}^k x - Qx \right\| \leq \frac{\varepsilon}{\|x\| + 1}.$$

Since $\bar{\tau}Qx = Qx$, it follows for any $j \in \mathbb{N}$ that

$$\left\| \frac{1}{n} \sum_{k=j}^{n+j-1} \bar{\tau}^k x - Qx \right\| = \left\| \bar{\tau}^j \left(\frac{1}{n} \sum_{k=0}^{n-1} \bar{\tau}^k x - Qx \right) \right\|$$
$$\leq \left\| \frac{1}{n} \sum_{k=0}^{n-1} \bar{\tau}^k x - Qx \right\| \leq \frac{\varepsilon}{\|x\| + 1},$$

so

$$\left| \left\langle x, \frac{1}{n} \sum_{k=j}^{n+j-1} \bar{\tau}^k x - Qx \right\rangle \right| \leq \|x\| \left\| \frac{1}{n} \sum_{k=j}^{n+j-1} \bar{\tau}^k x - Qx \right\| < \varepsilon.$$

Also, $\langle x, \iota(1) \rangle = \langle x, Q\iota(1) \rangle = \langle Qx, \iota(1) \rangle$ since $\bar{\tau}\iota(1) = \iota(\tau(1)) = \iota(1)$, so

$$\varphi(P)^2 = |\langle \iota(P), \iota(1) \rangle|^2 = |\langle x, \iota(1) \rangle|^2 \leq \|Qx\|^2 \|\iota(1)\|^2 = \langle x, Qx \rangle$$

since $\langle \iota(1), \iota(1) \rangle = \varphi(1^*1) = 1$. Thus

$$\left| \frac{1}{n} \sum_{k=j}^{n+j-1} \varphi(P\tau^k(P)) \right| = \left| \frac{1}{n} \sum_{k=j}^{n+j-1} \langle x, \bar{\tau}^k x \rangle \right| > |\langle x, Qx \rangle| - \varepsilon \geq \varphi(P)^2 - \varepsilon.$$

Since φ is a C*-trace, we have $\varphi(P\tau^k(P)) = \varphi((P\tau^k(P))^* P\tau^k(P)) \geq 0$ for all $k \in \mathbb{N}$, hence

$$\sum_{k=j}^{n+j-1} \varphi(P\tau^k(P)) > n(\varphi(P)^2 - \varepsilon).$$

This implies that $\varphi(P\tau^k(P)) > \varphi(P)^2 - \varepsilon$ for some $k \in \{j, j+1, \ldots, n+j-1\}$, i.e. E is relatively dense in $\mathbb{N}$. $\quad\square$

Recall that a finite factor is a finite von Neumann algebra $\mathfrak{M}$ that is also a factor, i.e. $\mathfrak{M} \cap \mathfrak{M}' = \mathbb{C}1$. In this case we can therefore take the trace of $\mathfrak{M}$ to be complex valued, so the conditions of Theorem 4.4 are satisfied when $\mathfrak{A}$ is a finite factor and φ is its trace.

We also mention that if in Theorem 4.4 we consider the special case where $\mathfrak{A}, \tau,$ and φ are taken as $B_\infty(\Sigma), \tau,$ and φ as defined above Definition 4.1, with $\mu(X) = 1$, then we get the usual measure theoretic theorem of Khintchine, namely, given any $\varepsilon > 0$, the set $\{k \in \mathbb{N} : \mu(S \cap T^{-k}(S)) > \mu(S)^2 - \varepsilon\}$ is relatively dense in $\mathbb{N}$ for all $S \in \Sigma$, where the condition $\mu(T^{-1}(S)) = \mu(S)$ can now be weakened to $\mu(T^{-1}(S)) \leq \mu(S)$. This is a stronger result than Poincaré's recurrence theorem (in the form stated above), despite the slightly weaker assumptions.

5. PHYSICAL INTERPRETATION

Consider a bounded quantum system $(\mathfrak{M}, \mathfrak{H}, H)$ and assume that $\mathfrak{M}$ is a factor. Let τ be the system's time-evolution, as in Proposition 3.3. Fix any $t > 0$. Since the trace tr of $\mathfrak{M}$ is faithful, Corollary 4.3 and Proposition 3.3 tell us that for any nonzero projection $P \in \mathfrak{M}$ there exists an $n(t) \in \mathbb{N}$ such that

$$\text{tr}\big(P\tau_{n(t)t}(P)\big) > 0. \tag{5}$$

Note that $\text{tr}(P\tau_{n(t)t}(P)) = \text{tr}(P\tau_{n(t)t}(P)P)$, which is similar to the form of ω' in Postulate 2.3, i.e. a state after a "yes" was obtained in a yes/no experiment with projection P. We now look at this similarity more closely by exploiting the analogy between quantum and classical mechanics described in Sections 2 and 3.

In Section 3 we saw that tr can be viewed as a quantum analogue of integration over a bounded set in phase space with respect to Lebesgue measure λ. To apply

Poincaré's recurrence theorem to classical mechanics, we know that the system in question has to confined to a bounded (Borel) set F in the phase space $\mathbb{R}^{2n}$, i.e. $\lambda(F) < \infty$. Since we can assume without loss that $\lambda(F) > 0$, we can normalize λ on F by defining a measure λ' on the Borel sets of $\mathbb{R}^{2n}$ by

$$\lambda'(S) = \lambda(S \cap F)/\lambda(F).$$

If we now view λ' as describing a state of the system (as explained in Section 2), then it essentially says that every part of F is equally likely to contain the state of the system (viewed as a point in the phase space). In other words, when we know nothing about the state of the system (aside from the fact that it is in F), then we can describe it by λ', or in C*-algebraic terms by the state φ on $B_\infty(\mathbb{R}^{2n})$ defined by

$$\varphi(g) = \int g \, d\lambda'.$$

Since $\mathrm{tr}(1) = 1$ and $\mathfrak{M}$ is a factor, tr is a state on $\mathfrak{M}$, and therefore we view tr as the quantum analogue of φ. By this analogy we would expect tr to describe the state of our quantum system when we know nothing about the system's state. This is indeed true in the special case where $\mathfrak{H}$ is finite dimensional and $\mathfrak{M} = \mathfrak{L}(\mathfrak{H})$, since for any rank one projection Q in $\mathfrak{M}$ we then have $\mathrm{tr}(Q) = 1/\dim(\mathfrak{H})$, which tells us that all values are equally probable when we measure an observable (assuming the observable has no degenerate eigenvalues). Furthermore, since tr is ultraweakly continuous, it is a normal state and hence it is given by a density operator (see Kadison and Ringrose, 1986), as one would expect for a physically meaningful state. We therefore suggest the following hypothesis:

Postulate 5.1. Consider a bounded quantum system $(\mathfrak{M}, \mathfrak{H}, H)$ where $\mathfrak{M}$ is a factor. When we have no information regarding the state of the system, the state is given by the trace tr of $\mathfrak{M}$.

So look at the case where we have no information about the state of our bounded quantum system. By Postulate 5.1 the state is then given by tr. At time 0 we perform a yes/no experiment with projection $P \in \mathfrak{M}$ on the system. Assuming the result is "yes," the state of the system after the experiment is given by the state ω on $\mathfrak{M}$ defined by

$$\omega(A) = \mathrm{tr}(PA)/\mathrm{tr}(P),$$

according to Postulate 2.3. (Also recall from Section 2 that the probability of getting "yes" is $\mathrm{tr}(P)$, therefore $\mathrm{tr}(P) > 0$ in this case.) By (5) we then have

$$p(t) := \omega\big(\tau_{n(t)t}(P)\big) > 0. \tag{6}$$

This simply tells us that if we were to repeat the yes/no experiment mentioned above exactly at the moment $n(t)t$, then there is a nonzero probability $p(t)$ that we

will again get "yes." By replacing t by $t' = n(t)t + 1$, we see that there is in fact an unbounded set of moments $n(t)t < n(t')t' < \cdots$ for which (6) holds.

So we have obtained a quantum mechanical version of recurrence. Note that the measure theoretic Poincaré recurrence theorem stated in Section 4 will give exactly the same result as (6), with the same physical interpretation, when applied to classical mechanics; just replace ω, tr, τ, and P by their classical analogues described in Section 2 and in this section. So we see that recurrence in quantum mechanics and in classical mechanics follow from the same theorem, namely Theorem 4.2, since Corollary 4.3 and measure theoretic Poincaré recurrence are both special cases of this theorem.

Of course, Theorem 4.4 tells us that for any $\varepsilon > 0$ there is in fact a relatively dense set M in $\mathbb{N}$ such that

$$\omega(\tau_{mt}(P)) > \text{tr}(P) - \varepsilon \tag{7}$$

for all $m \in M$. Since $\text{tr}(P)$ was the probability of getting a "yes" during the first execution of the yes/no experiment, we see from (7) that at the moments mt the probability of getting "yes" when doing the experiment a second time is larger or at least arbitrarily close to the original probability of getting "yes." Similar results concerning wave functions and density operators are presented in Hogg and Huberman (1982) and Percival (1961). If as before we replace ω, tr, τ, and P by their classical counterparts, and then apply Theorem 4.4 again, we find the same result as (7) for classical mechanics, with exactly the same interpretation as in quantum mechanics.

There is, however, a small technical problem: The probability of repeating the yes/no experiment exactly at the moment $n(t)t$ is zero. The same goes for any of the moments mt above. The next simple proposition remedies the situation in the quantum case:

Proposition 5.2. *Let τ be as in Proposition 3.3, where we take $\mathfrak{M}$ to be a finite factor. Then for any projection P in $\mathfrak{M}$, the mapping*

$$\mathbb{R} \to \mathbb{R} : t \mapsto tr(P\tau_t(P))$$

is continuous, where tr is the trace of $\mathfrak{M}$.

Proof: By Stone's theorem U_t in Proposition 3.3 is strongly continuous, so clearly the mapping $t \mapsto \tau_t(A)$ is weakly continuous for every $A \in \mathfrak{M}$. Hence $t \mapsto P\tau_t(P)$ is weakly continuous. We know that tr is ultraweakly continuous (see Kadison and Ringrose (1986), for example) and therefore it is weakly continuous on the unit ball. Since $\|P\tau_t(P)\| \leq 1$, we conclude that $t \mapsto \text{tr}(P\tau_t(P))$ is continuous. $\quad \square$

So from (7) we see that for every $m \in M$ there exists a $\delta_m > 0$ such that

$$\omega(\tau_s(P)) > \text{tr}(P) - \varepsilon \quad \text{for} \quad mt - \delta_m < s < mt + \delta_m.$$

This tells us that quantum mechanical recurrence is possible in practice, assuming we are working with a bounded quantum system as above, since there is a nonzero probability of repeating the yes/no experiment during one of the time-intervals $(mt - \delta_m, mt + \delta_m)$.

Of course, this remark leads to the next question: Which physical systems can be mathematically described as bounded quantum systems with the observable algebras being factors?

In classical mechanics, Poincaré's recurrence theorem applies to systems that are confined to a bounded set in phase space. From a physical standpoint this is true if the system is confined to a finite volume in space, and it is isolated from outside influences (which could increase its energy content), to prevent any of its momentum components to go to infinity. (To see this, use Cartesian coordinates. Here we assume that each potential of the form $-1/r$ or the like has some "cut-off" at small values of r, since, for example, particles are of finite size and collide when they get too close, the point being that there is not an infinite amount of potential energy available in the system.)

Most likely then (keeping in mind the close analogy between quantum and classical mechanics), recurrence will occur for quantum mechanical systems bounded in space and isolated from outside influences (apart from the yes/no experiments we perform on it). This is confirmed by Bocchieri and Loinger (1957) and Percival (1961). So we might guess that these types of systems can be described as bounded quantum systems in the sense of Definition 3.2 with $\mathfrak{M}$ a factor. This seems to be related to the nuclearity requirement in quantum field theory (see Haag, 1996), where a bounded set in classical phase space is intuitively thought of as corresponding to a finite dimensional subspace of the quantum state space. Since a quantum system whose state space $\mathfrak{H}$ is finite dimensional is clearly a bounded quantum system (since $\mathfrak{L}(\mathfrak{H})$ is a finite factor in this case), it certainly does not seem too far-fetched to conjecture that a physical system bounded in space and isolated from outside influences can be mathematically described as a bounded quantum system with a factor as the observable algebra. We will not pursue these matters further in this paper however.

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